BME I5000: Biomedical Imaging

Lecture 5
2D Reconstruction from Projections

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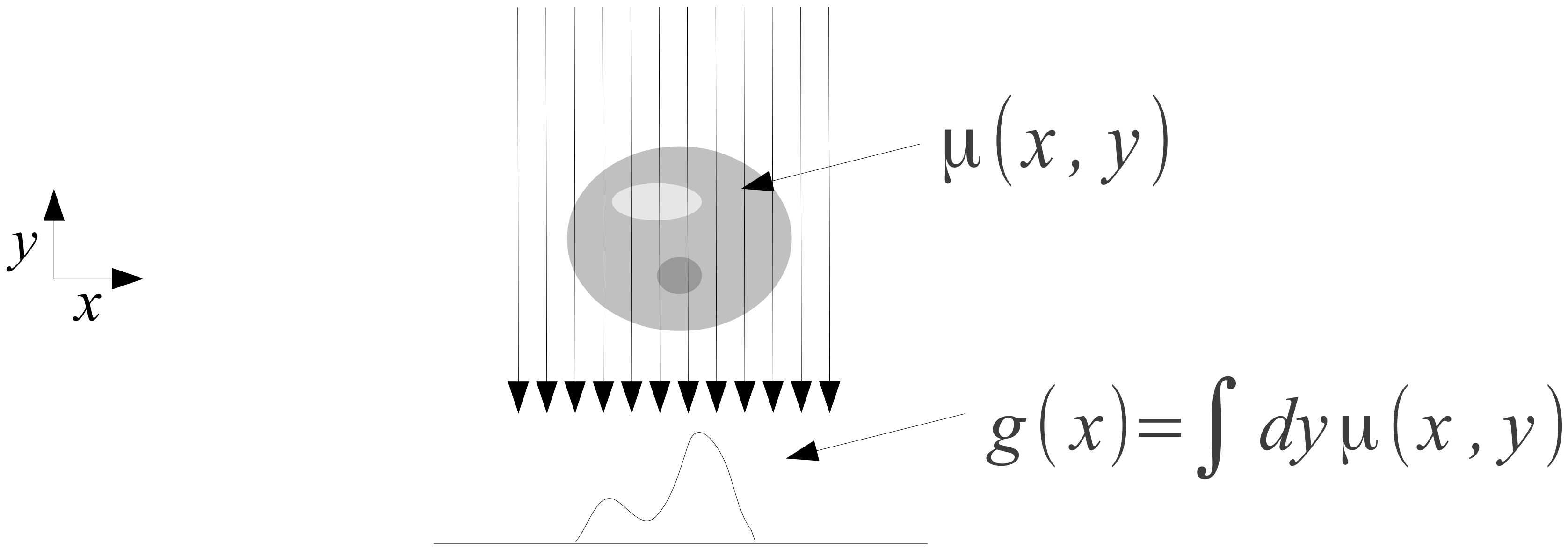
Blackboard: **http://cityonline.ccny.cuny.edu/**

Schedule

1. Introduction, Spatial Resolution, Intensity Resolution, Noise
2. X-Ray Imaging, Mammography, Angiography, Fluoroscopy
3. Intensity manipulations: Contrast Enhancement, Histogram Equalization
4. Computed Tomography
5. Image Reconstruction, Radon & Fourier Transform, Filtered Back Projection
6. Positron Emission Tomography
7. Maximum Likelihood Reconstruction
8. Magnetic Resonance Imaging
9. Fourier reconstruction, k-space, frequency and phase encoding
10. Optical imaging, Fluorescence, Microscopy, Confocal Imaging
11. Enhancement: Point Spread Function, Filtering, Sharpening, Wiener filter
12. Segmentation: Thresholding, Matched filter, Morphological operations
13. Pattern Recognition: Feature extraction, PCA, Wavelets
14. Pattern Recognition: Bayesian Inference, Linear classification

CT - Imaging Principle

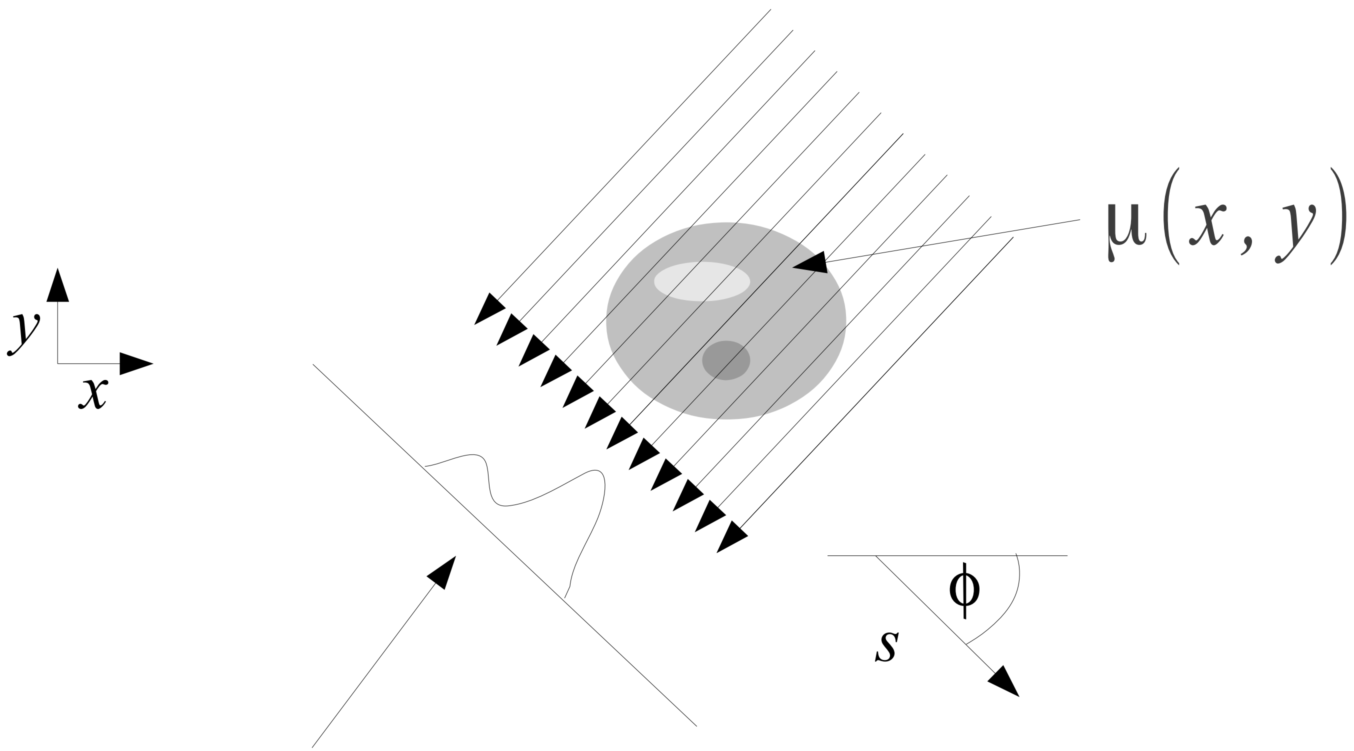
Computed Axial Tomography: Multiple x-ray projections are acquired around the object and a 2D image is computed from those projections.



Idea: Reconstruct 2D attenuation distribution $\mu(x,y)$ from multiple 1D x-ray projections $g(\ )$ taken at different angles ϕ .

CT – Forward Projection

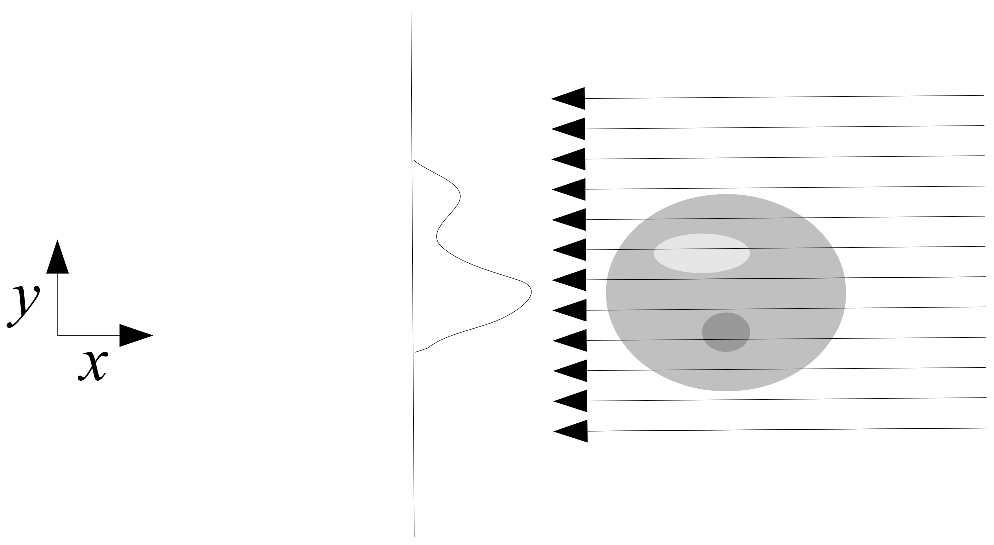
If we have more pixel we need more projections. To derive a general formalism for this linear inversion problem consider:



$$g(s,\phi)=\int_L dl\,\mu(x,y)$$

CT - Imaging Principle

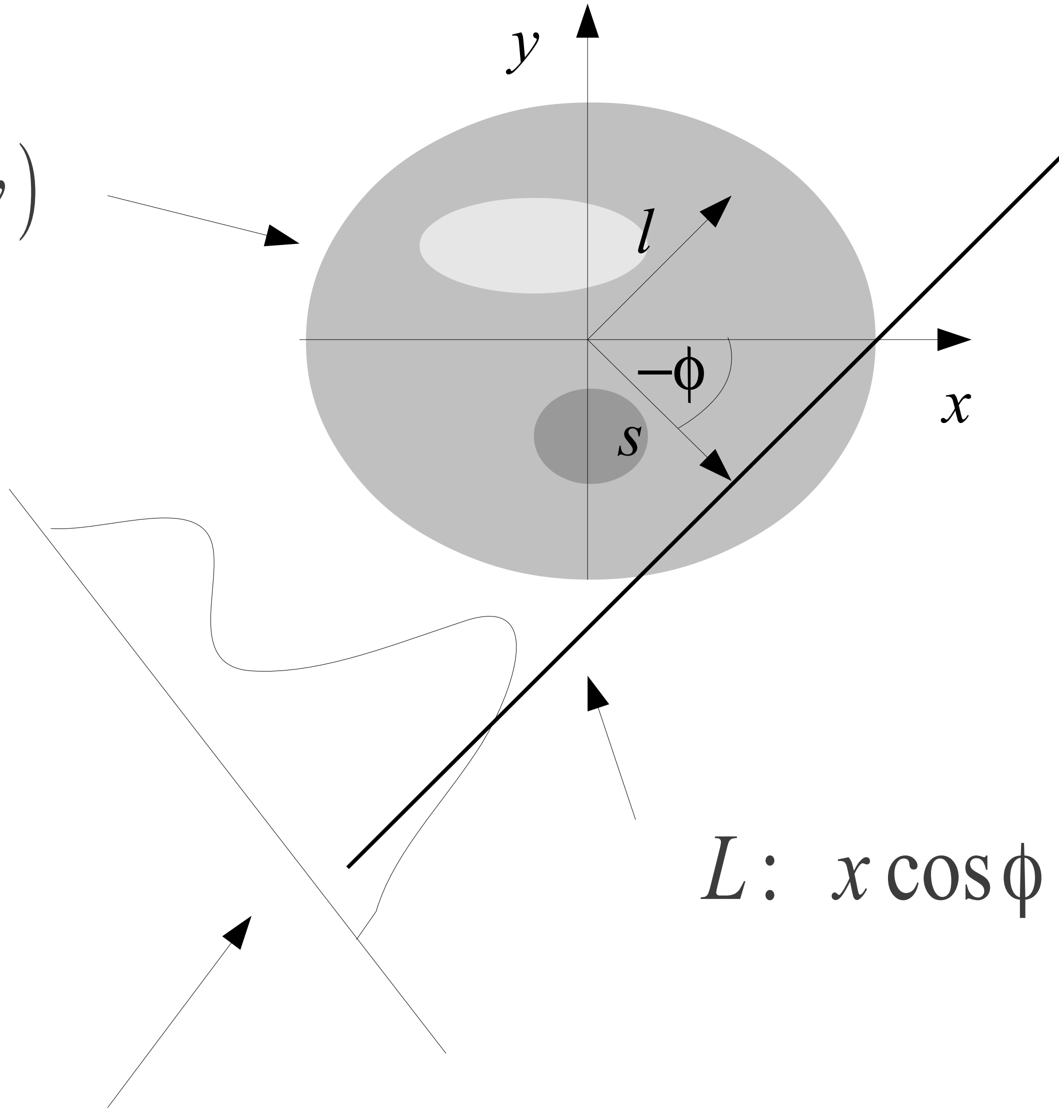
Computed Axial Tomography: Multiple x-ray projections are acquired around the object and a 2D image is computed from those projections.



Idea: Reconstruct 2D attenuation distribution $\mu(x,y)$ from multiple 1D x-ray projections $g(\ )$ taken at different angles ϕ .

CT – Radon Transform

$\mu(x,y)$



$L: \quad x\cos\phi + y\sin\phi = s$

$$g(\phi, s) = \int_L dl\, \mu(x,y)$$

CT – Radon Transform

This transformation from $\mu(x, y)$ to projections $g(s, \phi)$ is called a Radon Transform. It can be written in a number of ways:

$$g(s,\phi) = \int_L dl\, \mu(x,y)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx\, dy\, \mu(x,y)\, \delta(x \cos\phi + y \sin\phi - s)$$

As we will see next the **Radon transform in invertible**.
Therefore the idea of CT it to reconstruct $\mu(x, y)$ from $g(s, \phi)$:

x-ray imaging

$\mu(x,y)$ → $g(s,\phi)$

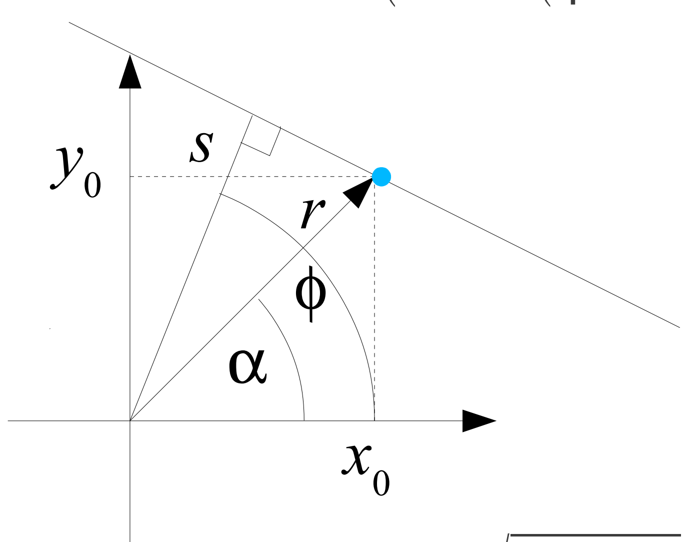
$\mu(x,y)$ ← $g(s,\phi)$

compute inverse
Radon transform

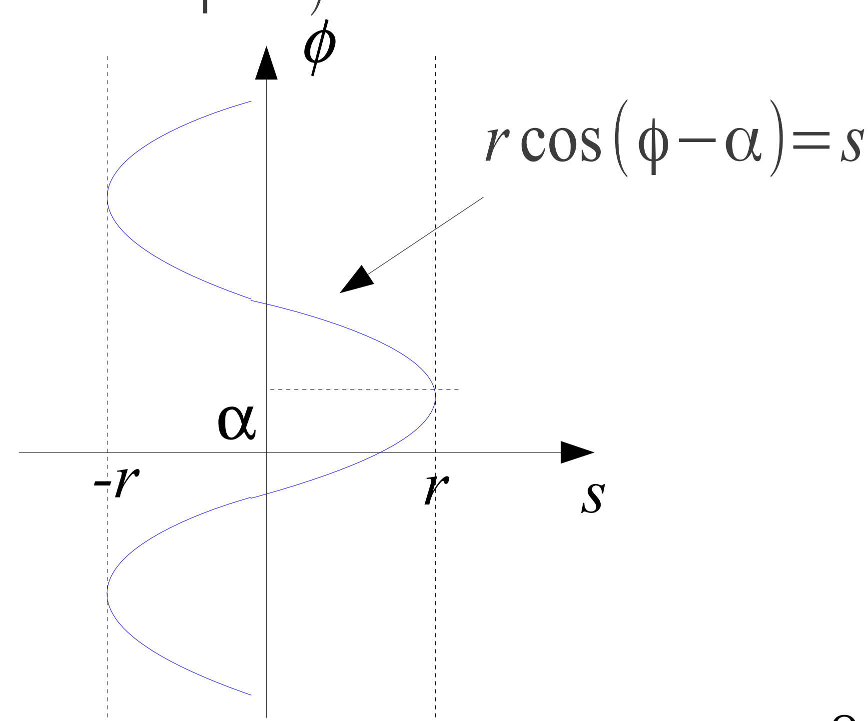
CT – Radon Transform

Note that: Each point in (s, ϕ) maps to a line in (x, y), and
Each point in (x, y) maps to a sinusoid in (s, ϕ).

$$g(s,\phi)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\,\delta(x-x_0,y-y_0)\,\delta(x\cos\phi+y\sin\phi-s)$$
$$=\delta(x_0\cos\phi+y_0\sin\phi-s)$$
$$=\delta(r\cos\alpha\cos\phi+r\sin\alpha\sin\phi-s)$$
$$=\delta(r\cos(\phi-\alpha)-s)$$



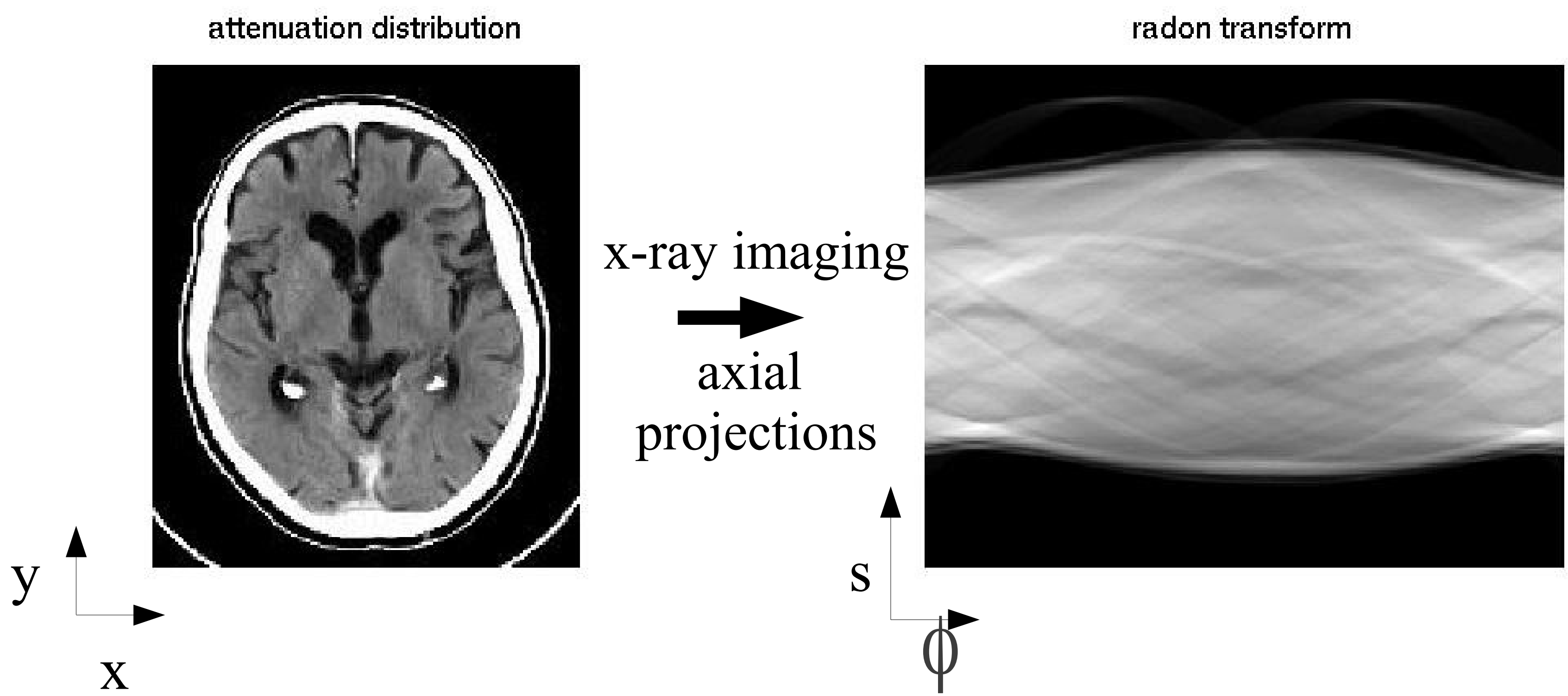
$$r=\sqrt{x^2+y^2}$$



CT – Radon Transform

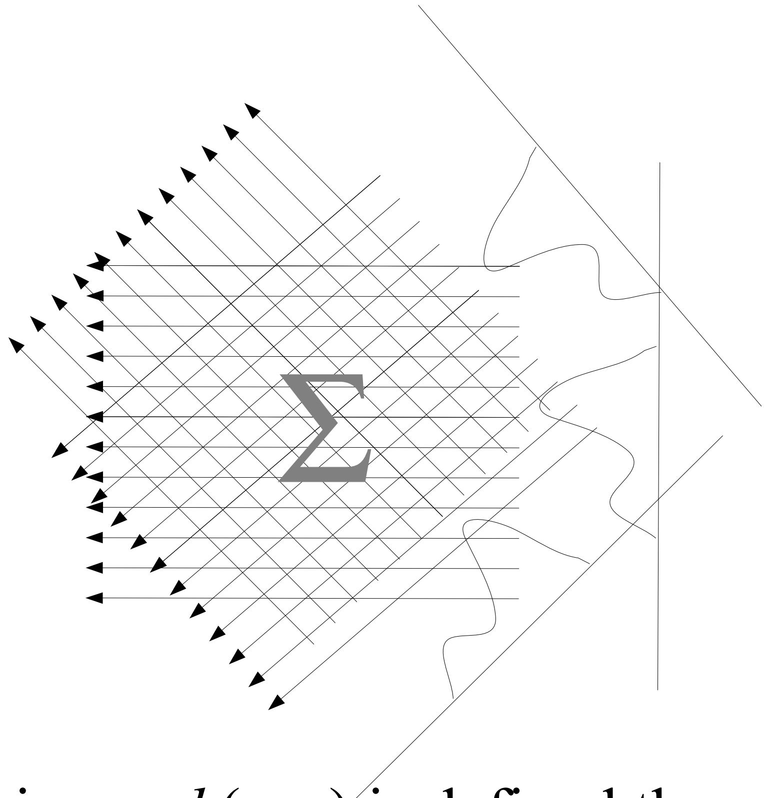
An image can be considered a linear superposition of points. The Radon transform is therefore a linear superposition of sinusoids:

$$g(s,\phi)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dy\, \mu(x,y)\,\delta(x\cos\phi+y\sin\phi-s)$$



CT – Back Projection

A naïve approach to reconstruction would be to sum the projection $g(s, \phi)$ back in the Cartesian 2D space (x, y)



Back-projections image $b(x, y)$ is defined then as

$$b(x,y)=\int_0^\pi d\phi\, g(x\cos\phi+y\sin\phi,\phi)$$

CT – Back Projection

In practice this integral needs to be evaluated numerically. This require 1D interpolation: Measurements $g(s, \phi)$ are only given for discrete angles $\phi_n = n\,\Delta\phi$ and discrete excentricities $s_m = m\,\Delta s$.

$$b(x, y) = \Delta\phi \sum_{n=1}^{\%N} g(x\cos\phi_n + y\sin\phi_n, \phi_n)$$

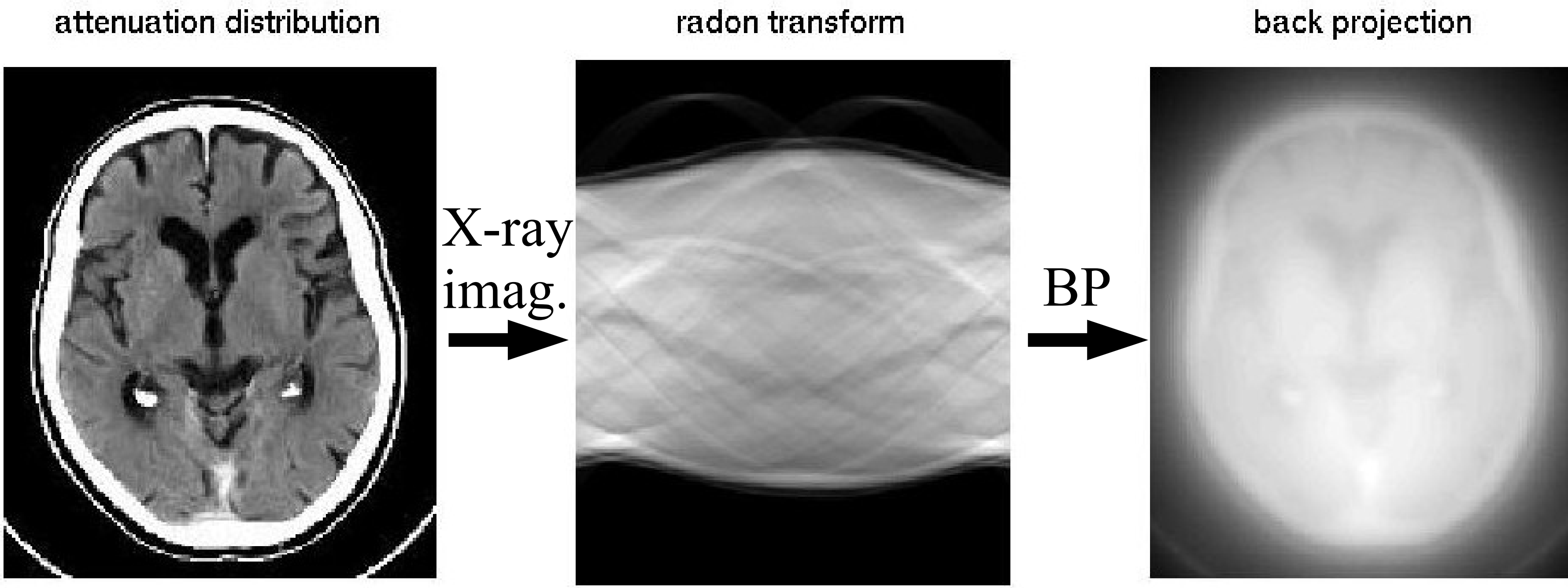
Values, $s = x\cos\phi_n + y\sin\phi_n$, at intermediate locations will be required and so $g(s, \phi_n)$ has to be interpolated from the values $g(s_m,\phi_n)$, $m=1, \ldots, M$ for a given ϕ_n.

Back-projection in MATLAB:

```
b = zeros(I,J);
[x,y] = meshgrid([1:J]-J/2,[1:I]-I/2);
for phi=0:179
   s = x*cos(pi/180*phi)+y*sin(pi/180*phi);
   b = b + interp1(sn,g(:,phi+1),s);
end
```

CT – Back Projection

Back-projection $b(x, y)$ gives a blurred version of original $\mu(x, y)$:



In fact, we can show that axial projection imaging followed by back projection is a LSI transform with $|r|^{-1}$ as PSF:

$$b(x, y) = \mu(x, y) * \frac{1}{|r|}$$

CT – Back Projection

Back-projection $b(x, y)$ gives a blurred version of original $\mu(x, y)$:

$$b(x,y)=\int_0^\pi d\phi\, g(x\cos\phi+y\sin\phi,\phi)$$

$$=\int_0^\pi d\phi \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'dy'\mu(x',y')\delta(x'\cos\phi+y'\sin\phi-x\cos\phi-y\sin\phi)$$

$$=\int_0^\pi d\phi \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'dy'\mu(x+x',y+y')\delta(x'\cos\phi+y'\sin\phi)$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'dy'\mu(x+x',y+y')\frac{1}{\sqrt{x'^2+y'^2}}$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'dy'\frac{1}{\sqrt{(x'-x)^2+(y'-y)^2}}\mu(x',y')=\frac{1}{|r|}*\mu(x,y)$$

Assignment 5: Using $x=r\cos\theta,\ y=r\sin\theta,\ \cos(\theta-\phi)=\ldots$ show that:

$$\int_0^\pi d\phi\,\delta(x\cos\phi+y\sin\phi)=\frac{1}{\sqrt{x^2+y^2}}$$

CT – Back Projection, Inverse Filtering

To recover $\mu(x, y)$ from the back-projections we have to invert this convolutions:

$$b(x,y) = \mu(x,y) * h(x,y)$$

with

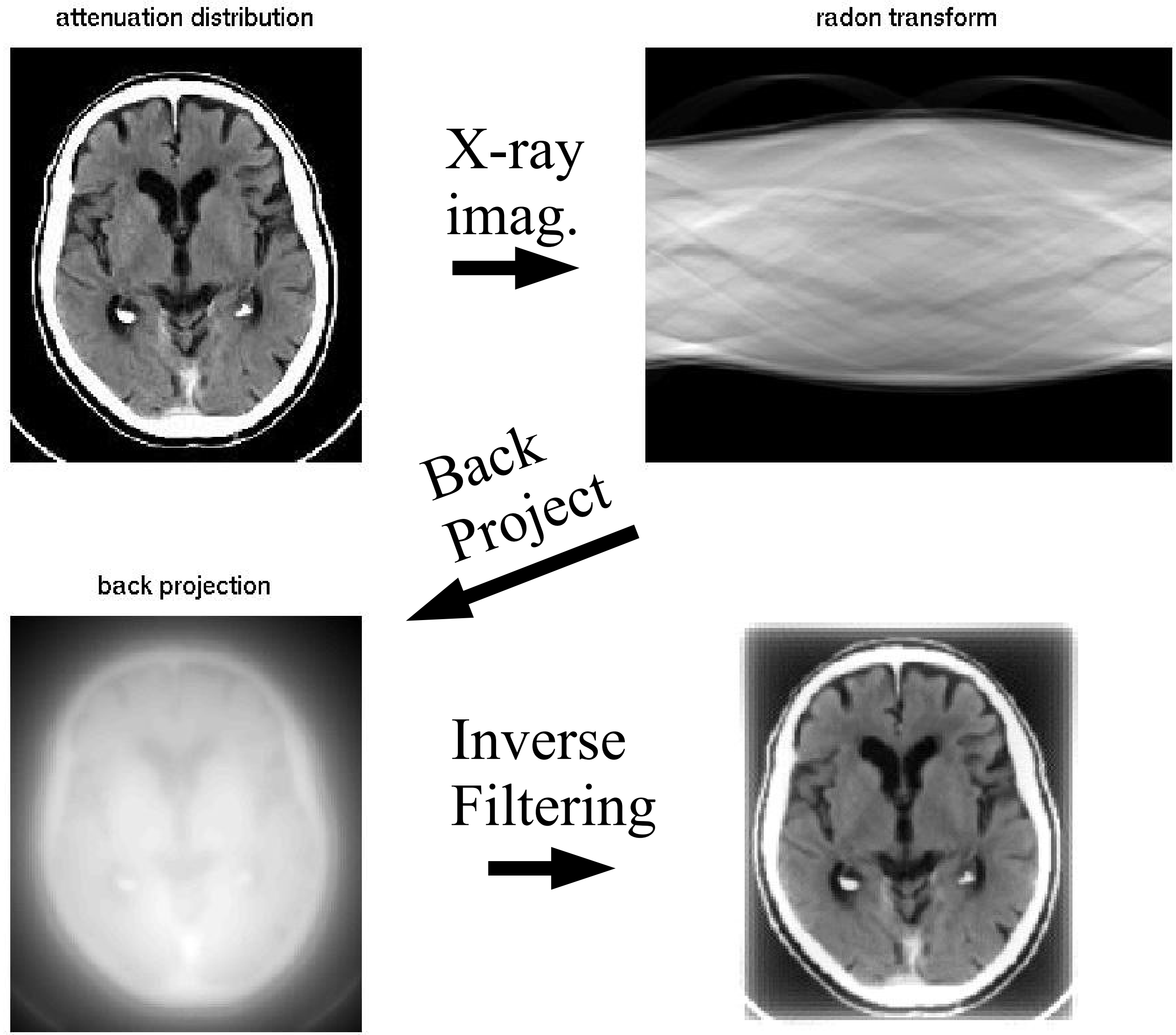
$$h(x,y) = \frac{1}{|r|} = \frac{1}{\sqrt{x^2+y^2}}$$

Note that convolutions can be inverted (under some assumptions)

$$\mu(x,y) = b(x,y) * h_{inv}(x,y)$$

To see this we have to introduce the **Fourier Transform** and the corresponding frequency domain **Convolution Theorem**.

CT – Back Projection, Inverse Filtering



Note edge effects

Fourier Transform (1D)

The **Fourier Transform** (FT) is defined as*

$$H(k) = FT[h(x)] = \int_{-\infty}^{\infty} dx\, h(x)\, e^{-i2\pi kx}$$

The FT is an invertible transformation

$$h(x) = FT^{-1}[H(k)] = \int_{-\infty}^{\infty} dk\, H(k)\, e^{i2\pi kx}$$

We can show this using $\int_{-\infty}^{\infty} dk\, e^{-i2\pi kx} = \delta(x)$

$$\int_{-\infty}^{\infty} dk\, H(k)\, e^{i2\pi kx} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk\, dx'\, h(x')\, e^{i2\pi k(x-x')} = h(x)$$

* Notational convention: Use *k* for spacial, and ω for temporal frequency.

Fourier Transform – Convolution Theorem

$$h(x)*g(x) \Leftrightarrow H(k)G(k)$$

Because the Fourier transform of the convolution ...

$$FT[h(x)*g(x)] = \int_{-\infty}^{\infty} dx\, h(x)*g(x)\, e^{-i2\pi kx} =$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx'\, h(x')\, g(x-x')\, e^{-i2\pi kx}$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx'\, h(x')\, g(x)\, e^{-i2\pi k(x+x')}$$

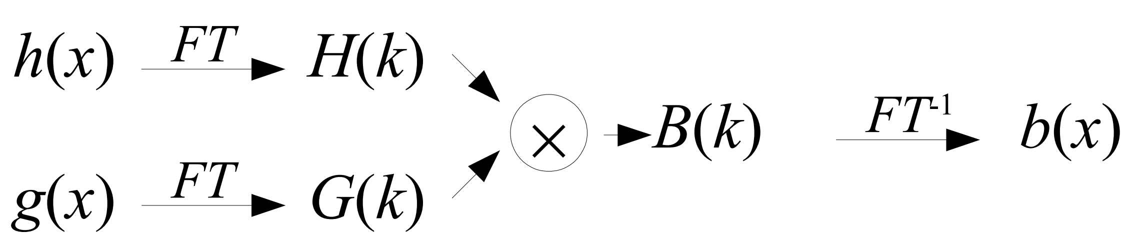
$$= \int_{-\infty}^{\infty} dx'\, h(x')\, e^{-i2\pi kx'} \int_{-\infty}^{\infty} dx\, g(x)\, e^{-i2\pi kx}$$

$$= H(k)G(k)$$

Fourier Transform – Convolution Theorem

Note that with the convolution theorem we can implement convolution as a multiplication in the frequency domain.

$$b(x) = g(x) * h(x) \Leftrightarrow B(k) = G(k)H(k)$$



Fourier Transform - Inverse Filter

With the Convolution Theorem we can derive the inverse convolution (or inverse filter)

$$b(x) = g(x) * h(x) \Leftrightarrow B(k) = G(k)\, H(k)$$

Therefore

$$G(k) = \frac{B(k)}{H(k)}$$

And the inverse filter is given by the inverse FT of $H^{-1}(k)$:

$$g(x) = FT^{-1}\left[\frac{1}{H(k)}\right] * b(x)$$

CT – Back Projection, Inverse Filtering

To recover $\mu(x, y)$ from the back projections we have to invert this convolutions:

$$b(x,y)=\mu(x,y)*h(x,y)$$

with

$$h(x,y)=\frac{1}{|\boldsymbol{r}|}=\frac{1}{\sqrt{x^2+y^2}}$$

Using

$$H(k_x,k_y)=FT\left[|\boldsymbol{r}|^{-1}\right]=|\boldsymbol{k}|^{-1}$$

And the Convolution Theorem we have:

$$\mu(x,y)=FT^{-1}\left[|\boldsymbol{k}|\right]*b(x,y)$$

Summary: To obtain the attenuation we have to back project the measurements $g(s, \phi)$ and filter that with $h^{-1}(x,y) = FT^{-1}[\ |\boldsymbol{k}|\ ]$.

Fourier Transform – FFT

The numerical implementation of the FT is discrete in x and k. It is referred to as Discrete Fourier Transform (DFT).

$$G[k]=\sum_{x=0}^{N-1} g[x]e^{-j2\pi kx/N}$$

$$g[x]=\frac{1}{N}\sum_{k=0}^{N-1} G[k]e^{j2\pi kx/N}$$

- A fast algorithm is available (FFT) to compute the DFT in only $N \log_2 N$ operations instead of N^2.
- Convolution with long filters is therefore often implemented using the FFT.
- FFT requires N to be a power of 2.

Demonstrate: `fft2, ifft2, fftshift`, inverse filtering, separability, Zero padding for power of 2 in FFT.

Fourier Transform – 2D and higher

The Fourier transform in 2D is defined correspondingly

$$H(k_x, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx\, dy\, h(x, y)\, e^{-i2\pi(k_x x + k_y y)}$$

Or in multiple dimensions with $\boldsymbol{k} = [k_x, k_y, k_z, \ldots]^T$, $\boldsymbol{r} = [x, y, z, \ldots]^T$

$$H(\boldsymbol{k}) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} d\boldsymbol{r}\, h(\boldsymbol{r})\, e^{-i2\pi \boldsymbol{k} \cdot \boldsymbol{r}}$$

The inverse transform is defined correspondingly.

Notice that the the Fourier transform can be applied **sequentially in each dimension**!

The Convolution Theorem applies in higher dimensions as well.

CT – Back Projection, Inverse Filtering

Summary: To obtain the attenuation we have to back project the measurements $g(s, \phi)$ and filter that with $h^{-1}(x,y) = FT^{-1}[\ |\boldsymbol{k}|\ ]$.

The problem with this approach:

1. The singularity at $|\boldsymbol{k}|=0$ may become a problem (total energy=0)
2. The area of support for $b(x,y)$ needs to be large. This is why we have errors at the edges.

An alternative method that overcomes these problems is to filter first and then back-project.

This will require that we introduce the Projection Slice theorem.

CT – Projection Slice Theorem

This theorem is used to invert the Radon transform. It relates the Randon transform $g(s,\phi)$ of a function $\mu(x,y)$ with 2D Fourier transform of that function.

Denote the 2D Fourier transform of $\mu(x,y)$ with

$$F(k_x, k_y) = FT_{x,y}[\mu(x, y)]$$

And denote the 1D Fourier transform of the Radon transform as

$$G(k, \phi) = FT_s[g(s, \phi)]$$

The Projection Slice Theorem states then

$$G(k, \phi) = F(k \cos\phi, k \sin\phi)$$

CT – Filter Back Projection

The original image $\mu(x,y)$ by definition:

$$\mu(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\,F(k_x,k_y)\,e^{i2\pi(k_x x+k_y y)}$$

or in polar coordinates:

$$\mu(x,y)=\int_{0}^{2\pi}\int_{0}^{\infty} d\phi\,dk\,k\,F(k\cos\phi,k\sin\phi)\,e^{i2\pi k(x\cos\phi+y\sin\phi)}$$

$$=\int_{0}^{\pi}\int_{-\infty}^{\infty} d\phi\,dk\,|k|\,F(k\cos\phi,k\sin\phi)\,e^{i2\pi k(x\cos\phi+y\sin\phi)}$$

$$=\int_{0}^{\pi}\int_{-\infty}^{\infty} d\phi\,dk\,|k|\,G(k,\phi)\,e^{i2\pi k(x\cos\phi+y\sin\phi)}$$

CT – Filter Back Projection

By defining the following filtered Radon transform

$$\hat{g}(s,\phi)=\int_{-\infty}^{\infty} dk\,|k|\,G(k,\phi)\,e^{i2\pi ks}$$

The previous equation becomes a simple back projection

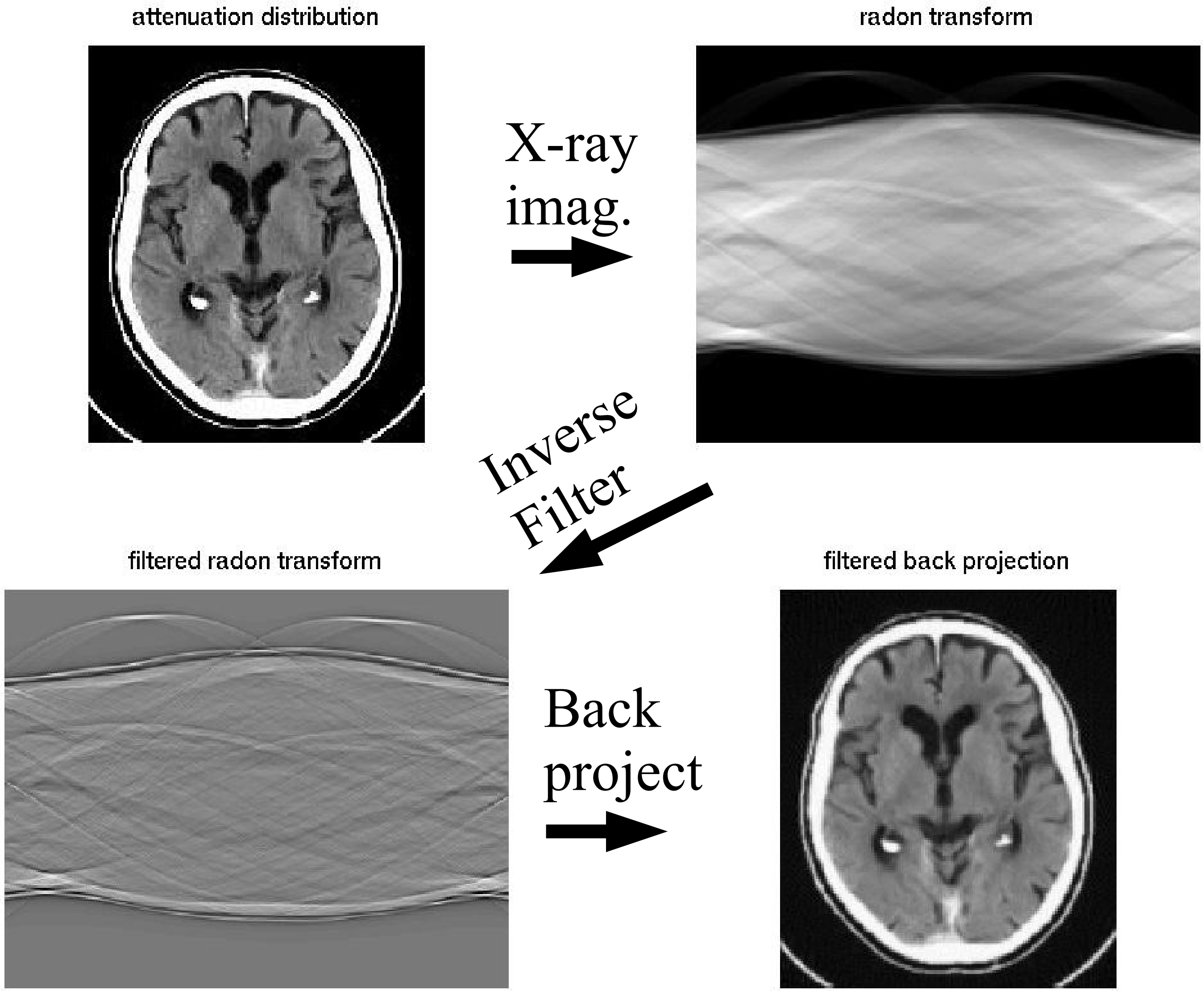
$$\mu(x,y)=\int_{0}^{\pi} d\phi\,\hat{g}(x\cos\phi+y\sin\phi,\phi)$$

Notice that the first operation is 1D filtering of the Radon transform along the excentricity axis r with $H(k) = |k|$.

$$\hat{g}(s,\phi)=FT^{-1}[|k|]*g(s,\phi)$$

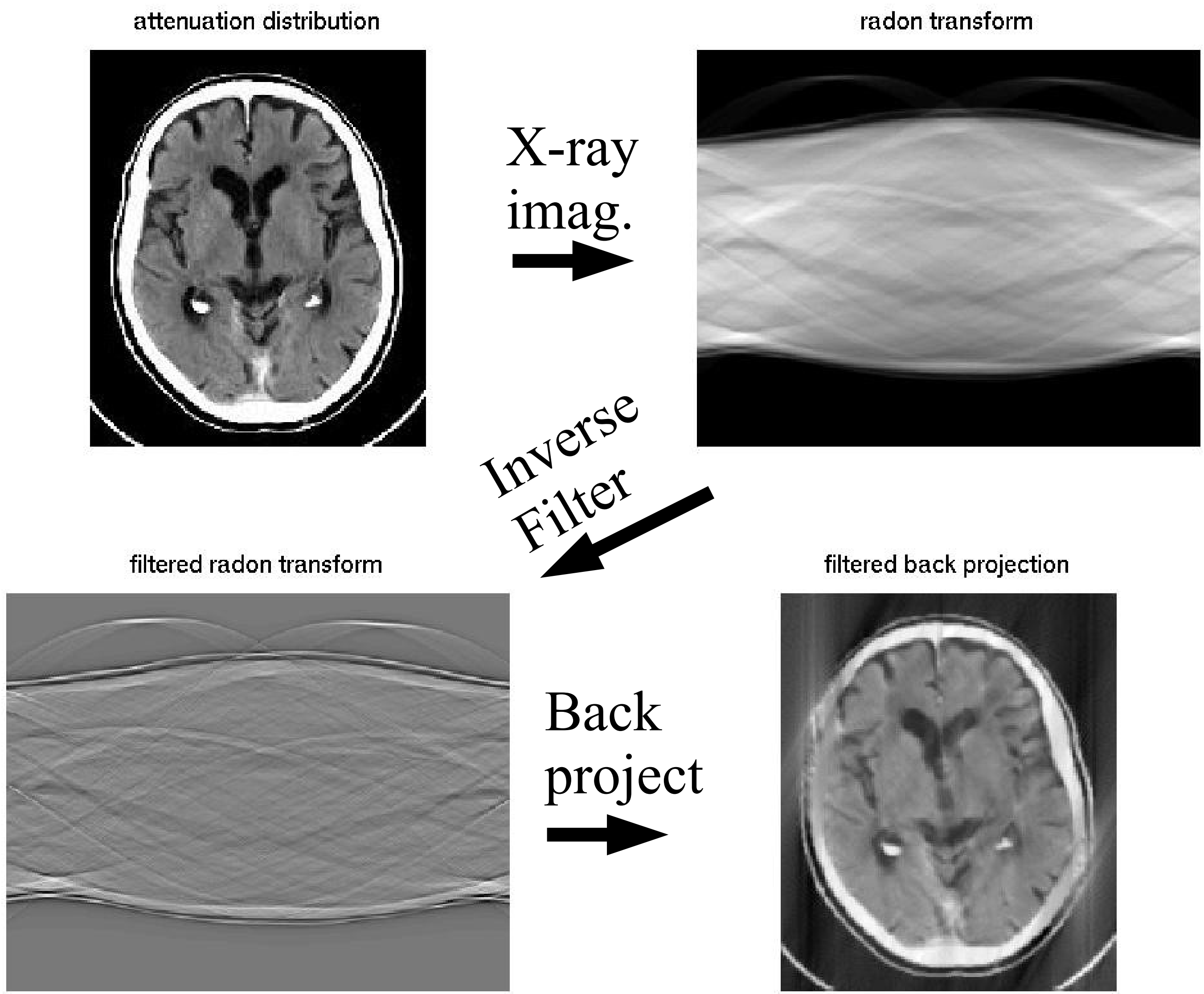
Summary: To obtain the attenuation we have to filter $g(s, \phi)$ with $h^{-1}(s) = FT^{-1}[\,|k|\,]$ and back-project the result.

CT – Filter Back Projection



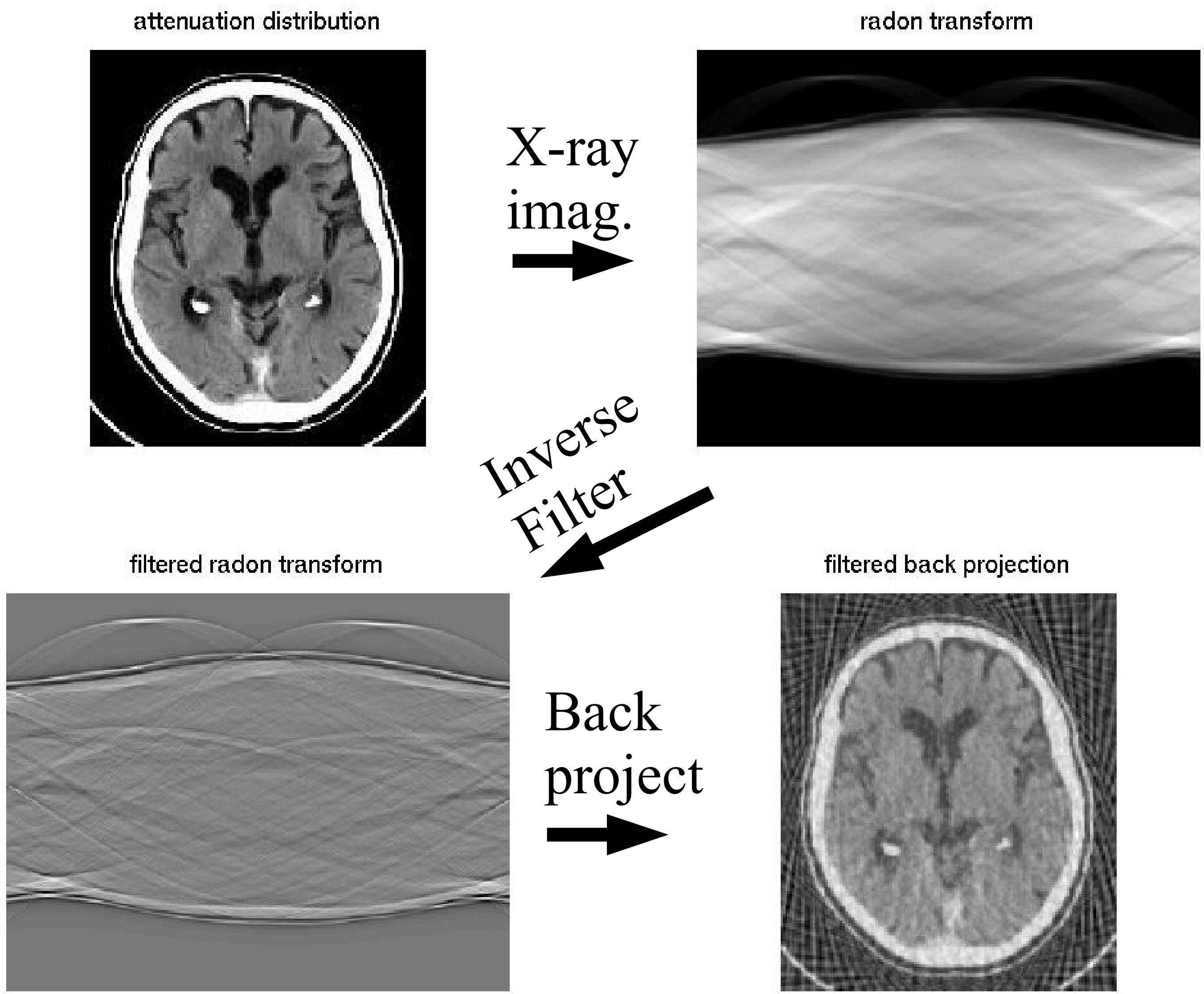
Demo: limited view angles, limited angular sampling, noise.

CT – Filter Back Projection



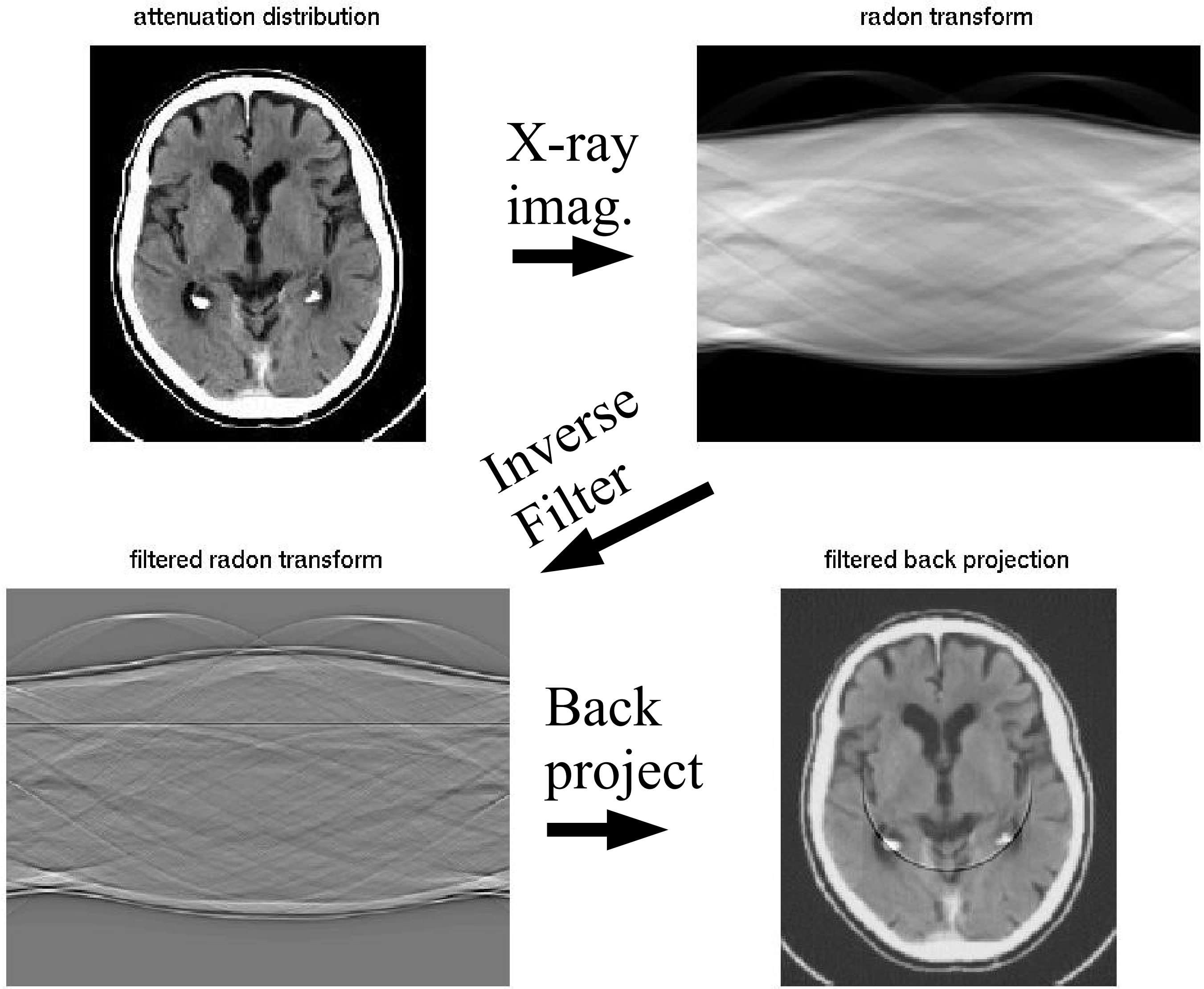
Limited view angles: 150°

CT – Filter Back Projection



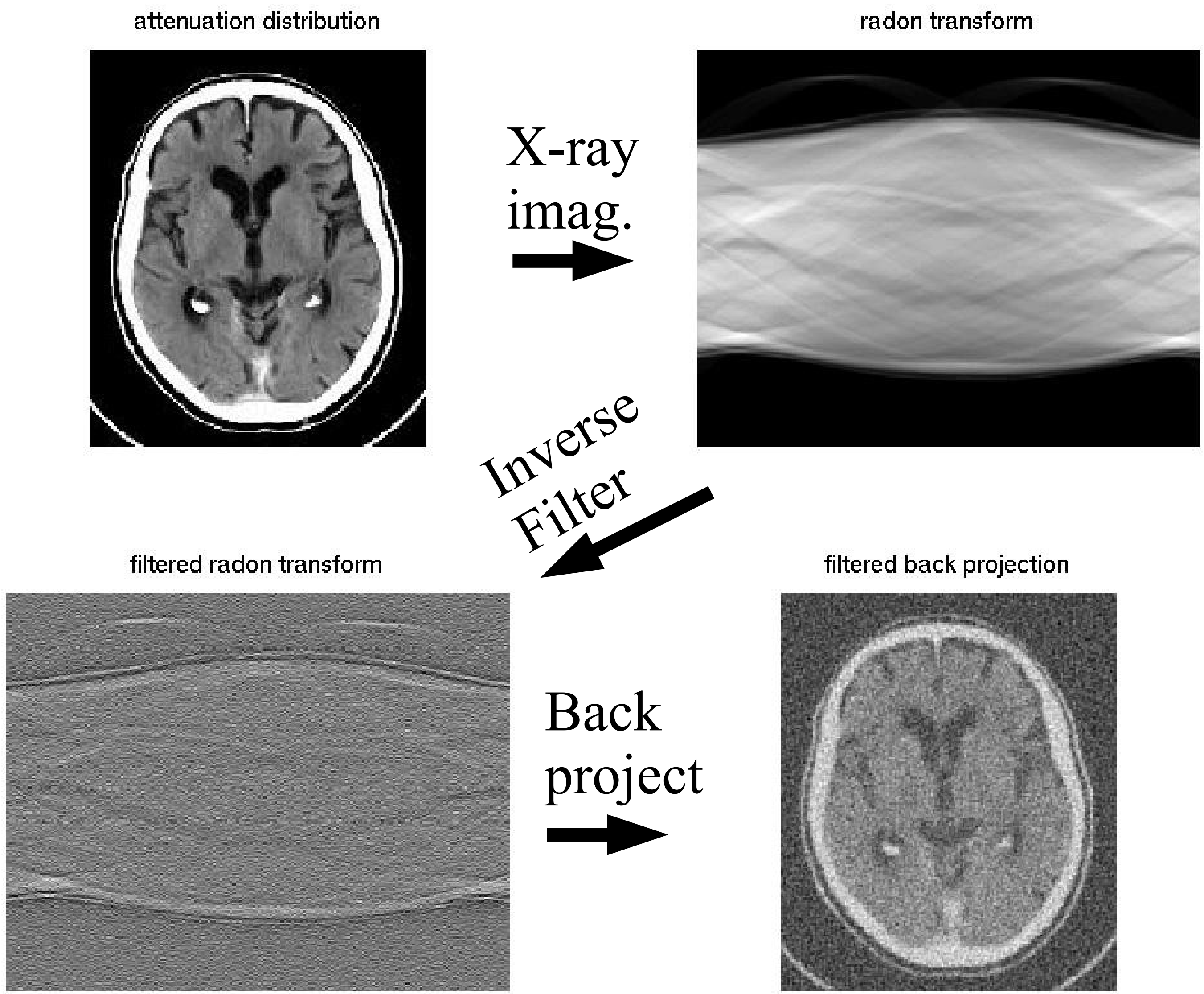
Limited angular sampling: 5° steps

CT – Filter Back Projection



Detector calibration error: 10% at s = 80

CT – Filter Back Projection



Sensor noise: additive Gaussian with 10% of std of *g*()

CT – Filter Back Projection

Assignment 6: Implement Filter Back Projection.

1. Simulate axial x-ray imaging using the matlab radon function
2. Filter with the Radon transform with inverse $H(k) = |k|$ using the 1D Fourier transform along the s axis. (Note that you may need zero padding in the FFT).
3. Back project the filtered Radon transform.
4. Display the original and the result of each operation as in previous slide.