BME I5000: Biomedical Imaging

Lecture FT
Fourier Transform Review

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Blackboard: http://cityonline.ccny.cuny.edu/

Fourier Transform (1D)

The **Fourier Transform** (FT) is defined as*

$$H(k) = FT[h(x)] = \int_{-\infty}^{\infty} dx\, h(x)\, e^{-i 2\pi kx}$$

The FT is an invertible transformation

$$h(x) = FT^{-1}[H(k)] = \int_{-\infty}^{\infty} dk\, H(k)\, e^{i 2\pi kx}$$

We can show this using $\int_{-\infty}^{\infty} dk\, e^{-i 2\pi kx} = \delta(x)$

$$\int_{-\infty}^{\infty} dk\, H(k)\, e^{i 2\pi kx} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk\, dx'\, h(x')\, e^{-i 2\pi k(x'-x)} = h(x)$$

* Notational convention: Use *k* for spacial, and ω for temporal frequency.

Fourier Transform – Frequency Domain

H(*k*) is in general complex valued

$$H(k) = R(k) + i\,I(k) = A(k)\,e^{i\phi(k)}$$

with *R*(*k*) = Re(H(k)), *I*(*k*) = Im(*H*(*k*)) and

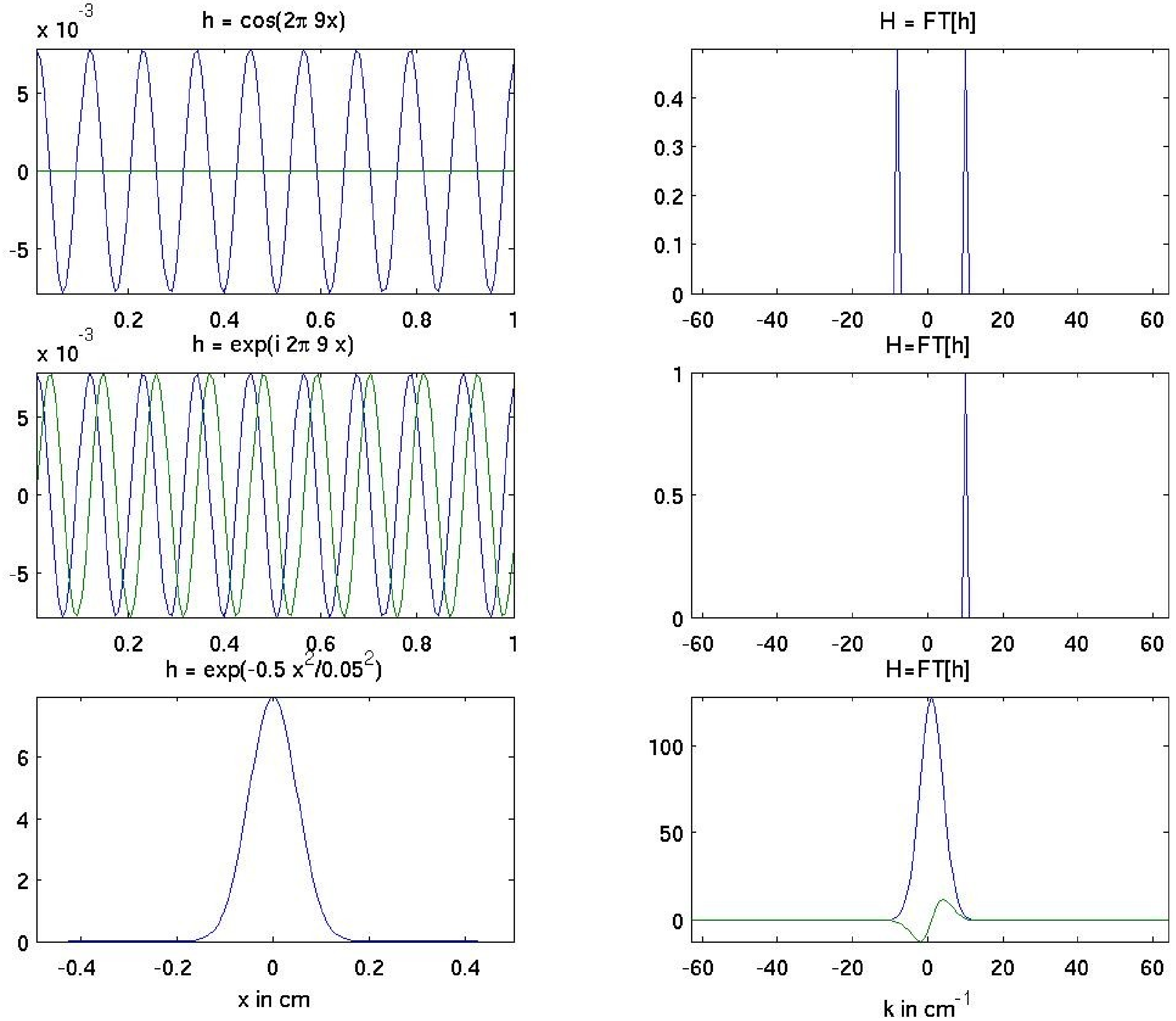
Amplitude: $A(k) = |H(k)| = \sqrt{R(k)^2 + I(k)^2}$

Phase: $\phi(k) = \arctan\left(\frac{I(k)}{R(k)}\right)$

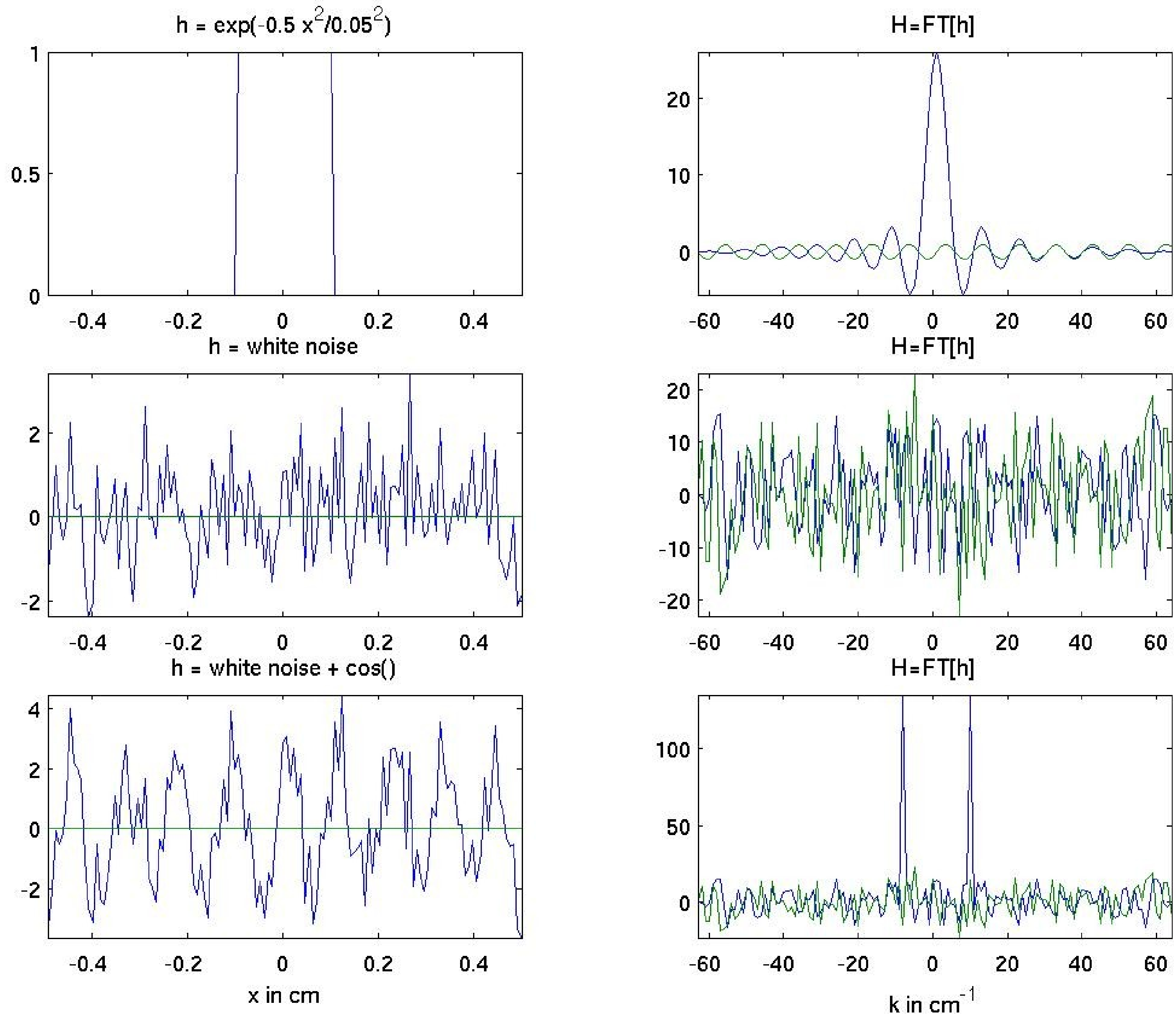
Positive frequencies *k*>0: counterclockwise rotation
Negative frequencies *k*<0: clockwise rotation

$$H(k) = \delta(k - k_0), \qquad h(x) = e^{i 2\pi k_0 x}$$

Fourier Transform - Examples



Fourier Transform - Examples



Fourier Transform – Convolution Theorem

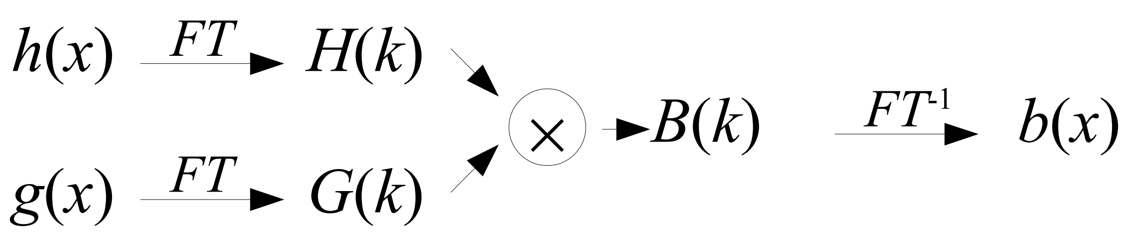
Convolution is defined as

$$b(x) = \int_{-\infty}^{\infty} dx' \, h(x') \, g(x - x') = h(x) * g(x)$$

Convolution Theorem states that

$$B(k) = FT[h(x) * g(x)] = H(k) G(k)$$

Note that with the convolution theorem we can implement convolution as a multiplication in the frequency domain.



Fourier Transform – Convolution Theorem

Proof

$$FT\left[h(x)*g(x)\right]=\int_{-\infty}^{\infty} dx\, h(x)*g(x)\, e^{-i2\pi kx}=$$

$$=\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx'\, h(x')\, g(x-x')\, e^{-i2\pi kx}$$

$$=\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx'\, h(x')\, g(x)\, e^{-i2\pi k(x+x')}$$

$$=\int_{-\infty}^{\infty} dx'\, h(x')\, e^{-i2\pi kx'} \int_{-\infty}^{\infty} dx\, g(x)\, e^{-i2\pi kx}$$

$$=H(k)\,G(k)$$

Fourier Transform - Inverse Filter

With the Convolution Theorem we can derive the inverse convolution (or inverse filter)

$$b(x) = g(x) * h(x) \Leftrightarrow B(k) = G(k)H(k)$$

Therefore

$$G(k) = \frac{B(k)}{H(k)}$$

And the inverse filter is given by the inverse FT of $H^{-1}(k)$:

$$g(x) = FT^{-1}\left[\frac{1}{H(k)}\right] * b(x)$$

Fourier Transform – 2D and higher

The Fourier transform in 2D is defined as

$$H(k_x, k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dy\, h(x, y)\, e^{-i2\pi(k_x x + k_y y)}$$

The inverse transform is defined correspondingly.

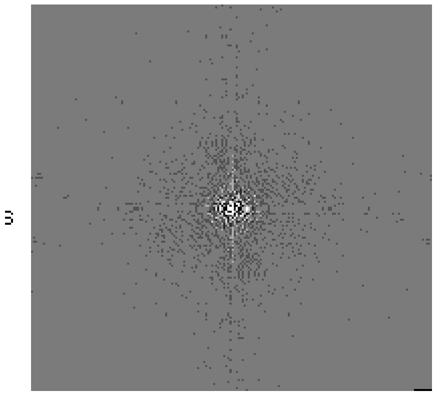
Note that the exponent is **separable** and therefor the Fourier transform can be applied **sequentially in each dimension**!

$$H(k_x, k_y) = \int_{-\infty}^{\infty} dy\, e^{-i2\pi k_y y} \int_{-\infty}^{\infty} dx\, h(x, y)\, e^{-i2\pi k_x x}$$

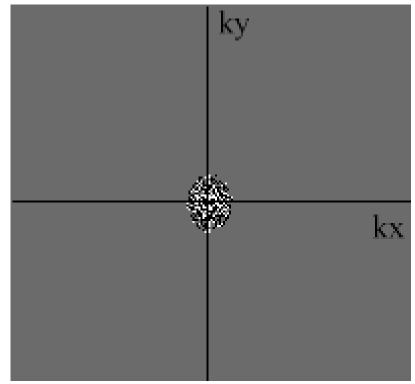
In multiple dimensions with $\boldsymbol{k} = [k_x, k_y, k_z, \ldots]^T$, $\boldsymbol{r} = [x, y, z, \ldots]^T$

$$H(\boldsymbol{k}) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} d\boldsymbol{r}\, h(\boldsymbol{r})\, e^{-i2\pi \boldsymbol{k}\cdot\boldsymbol{r}}$$

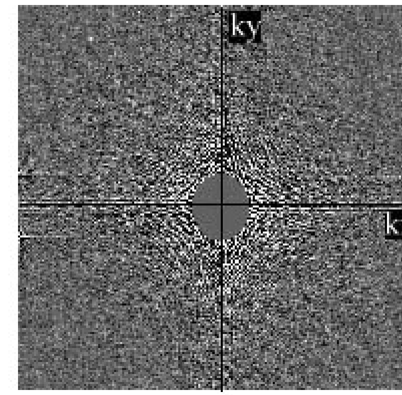
Fourie Transform k-space



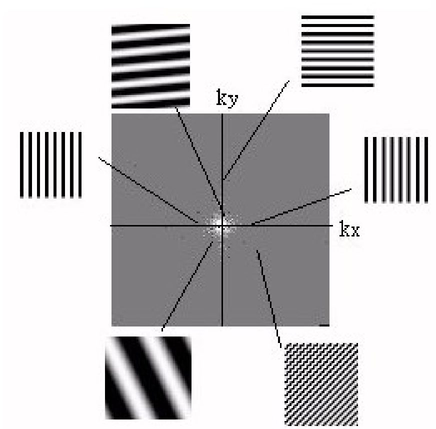
The Mona Lisa in k-Space



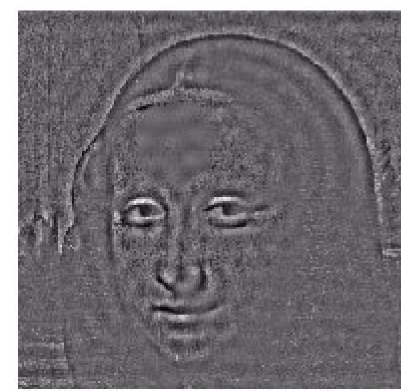
k-Space



k-Space



Low Frequency Mona

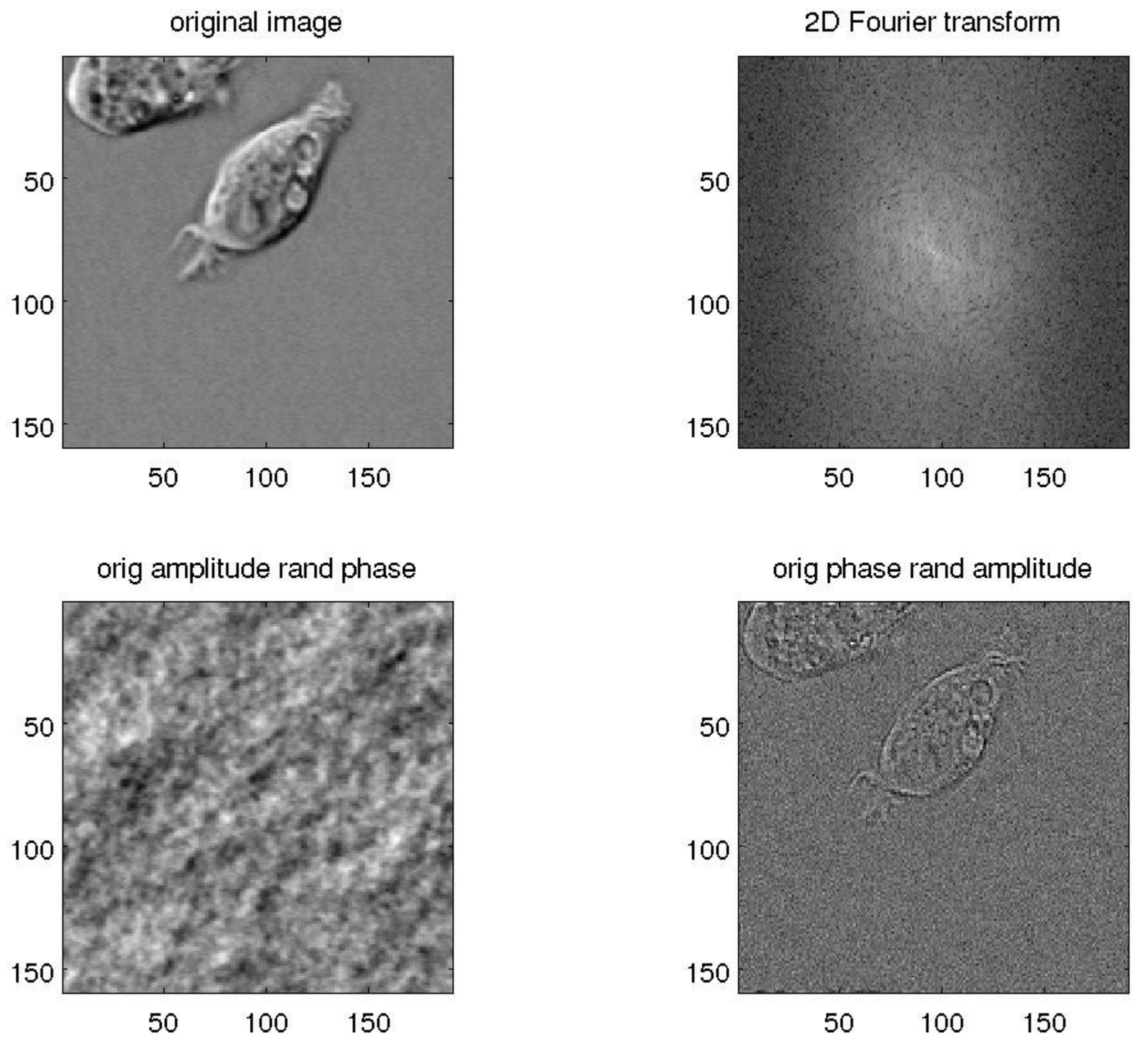


High Frequency Mona

Source: (C.A. Mistretta)

2D Fourier Transform – Phase and Amplitude

Note that the in images the phase carries most of the information



2D Fourier Transform – Phase and Amplitude

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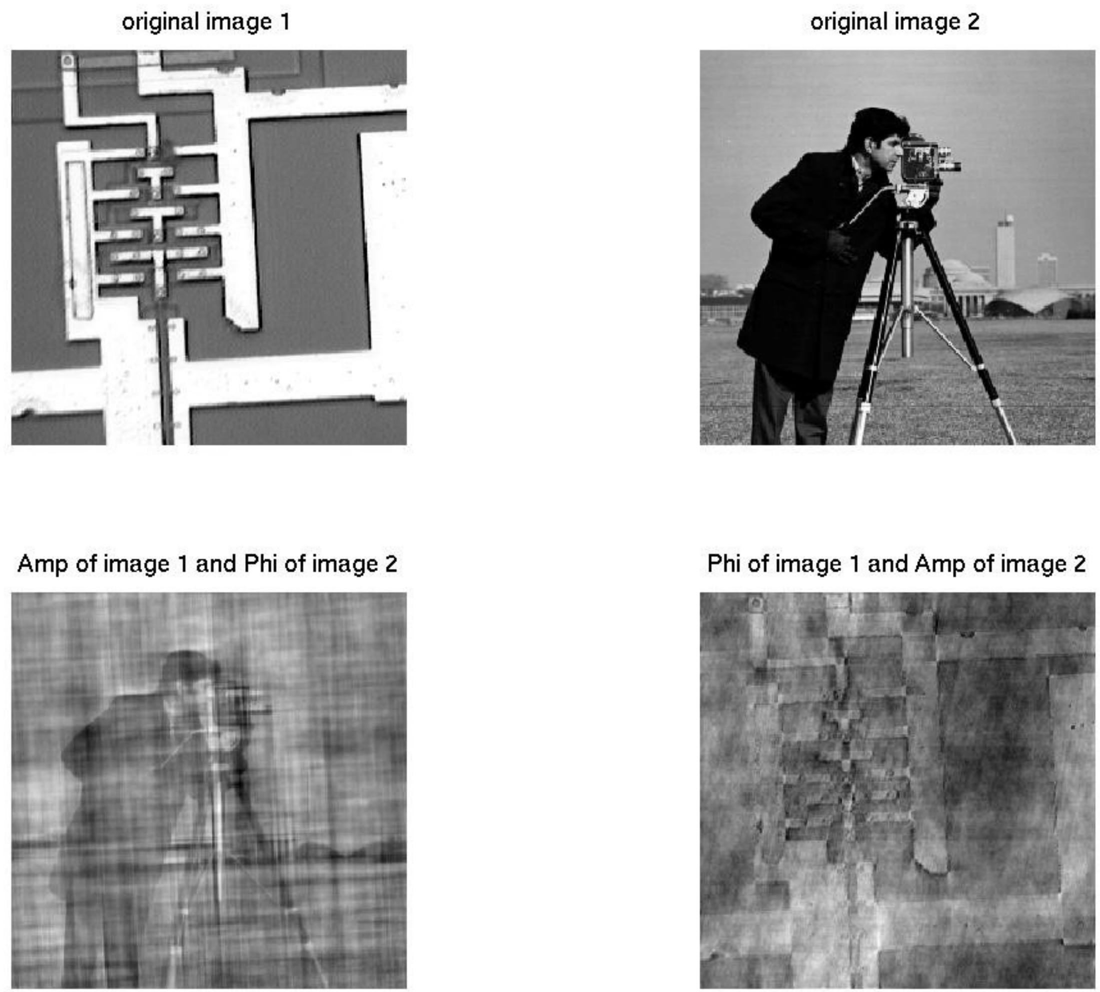


Fourier Transform – 2D Convolution

The 2D convolution with a PSF $h(x,y)$ is

$$b(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'dy' h(x',y') g(x-x',y-y')$$
$$=h(x,y)*g(x,y)$$

The Convolution Theorem applies in higher dimensions as well.

$$B(k_x,k_y)=G(k_x,k_y)H(k_x,k_y)$$

Even though $h(x,y)$ may not be separable ($h(x,y) \neq h(x)*h(y)$) with the convolution theorem we never have to truly compute a 2D convolution.

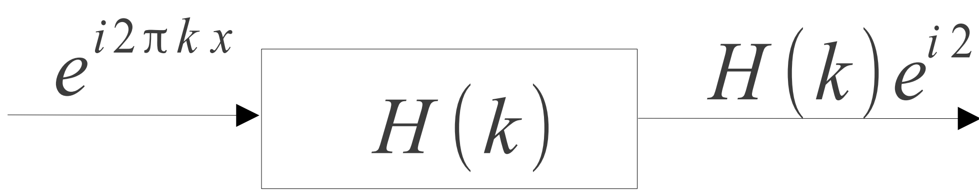
Fourier Domain System Response

Consider stationary oscillatory input to a LSI system $h(x)$:

$$g(x) = e^{i2\pi kx}$$

$$b(x) = h(x) * g(x) = \int_{-\infty}^{\infty} dx'\, h(x')\, e^{i2\pi k(x-x')} = H(k)\, e^{i2\pi kx}$$

The output is the input times the FT of the impulse response



The oscillation with frequency k has been modified in phase ϕ and amplitude A

$$A = |H(k)| \qquad \phi = arg(H(k))$$

$$H(k) = A e^{i\phi}$$

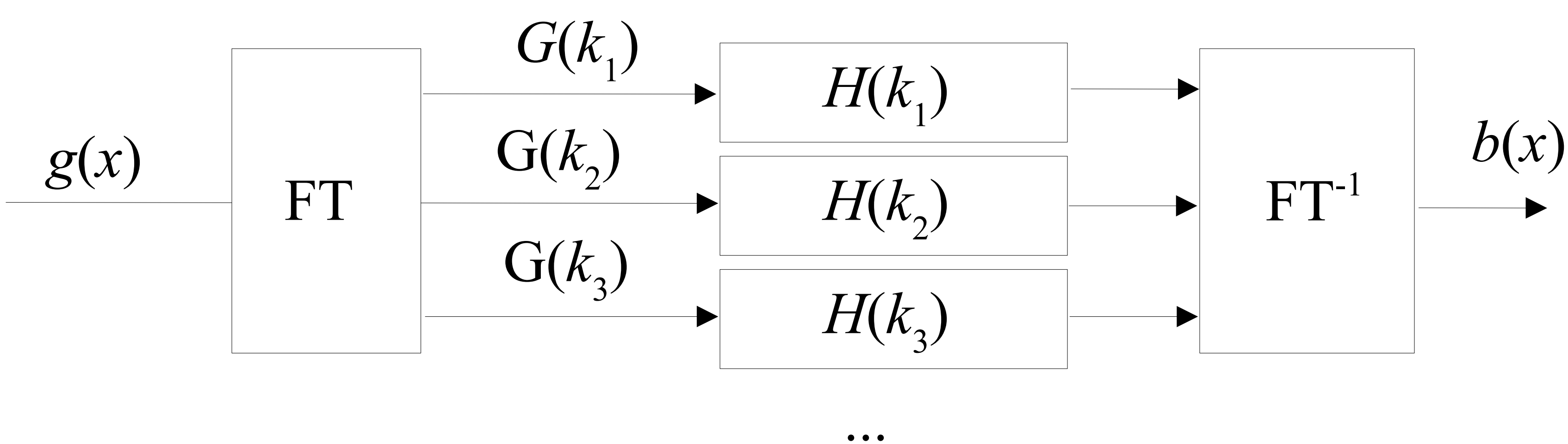
Fourier Domain System Response

FT inversion formula tells us that arbitrary input g(x) can be decomposed into sum of oscillations weighted by $G(k)$

$$g(x) = \int_{-\infty}^{\infty} dk\, G(k)\, e^{i2\pi kx}$$

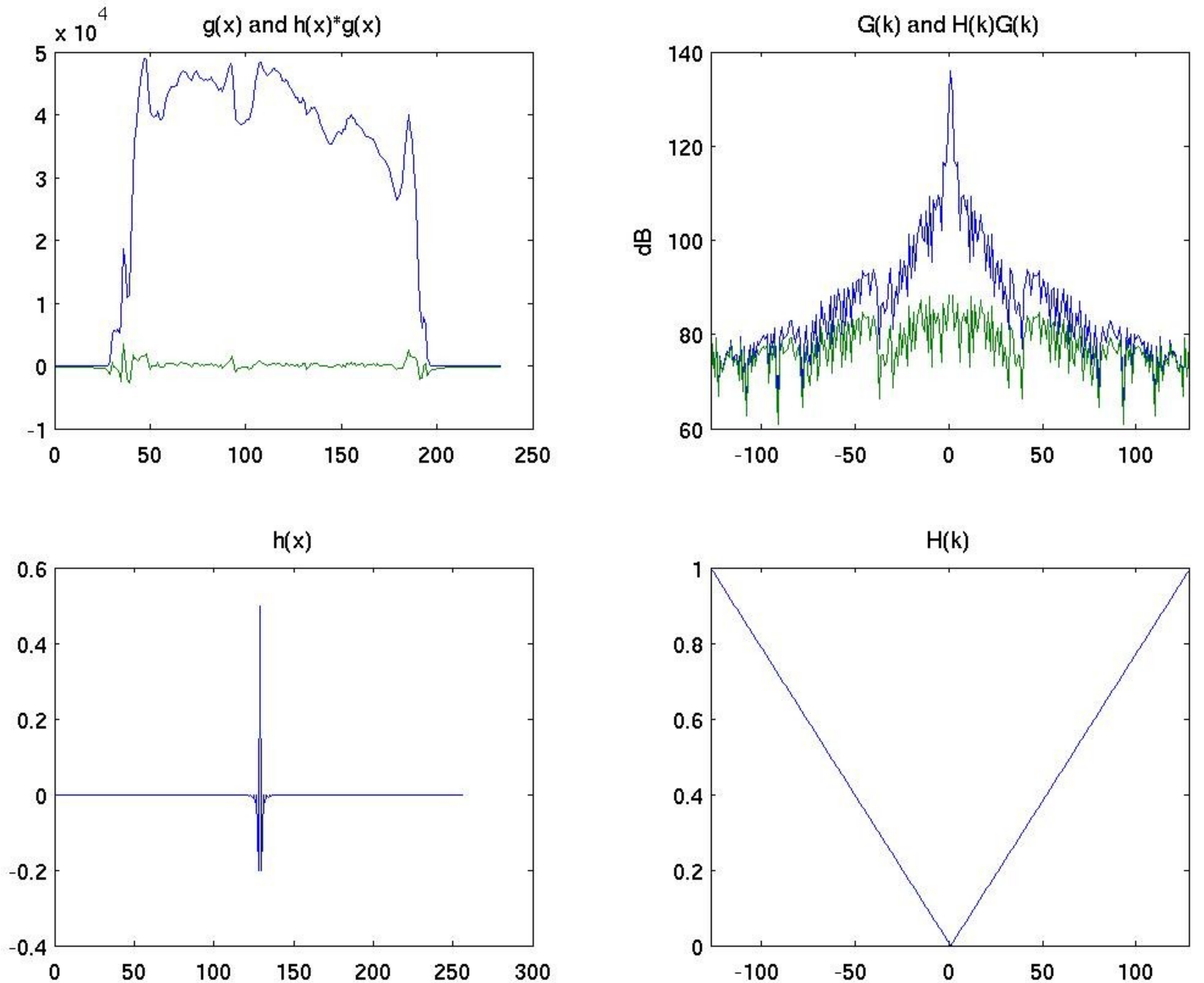
The system response to that is given by the convolution theorem

$$B(k) = H(k)\, G(k)$$



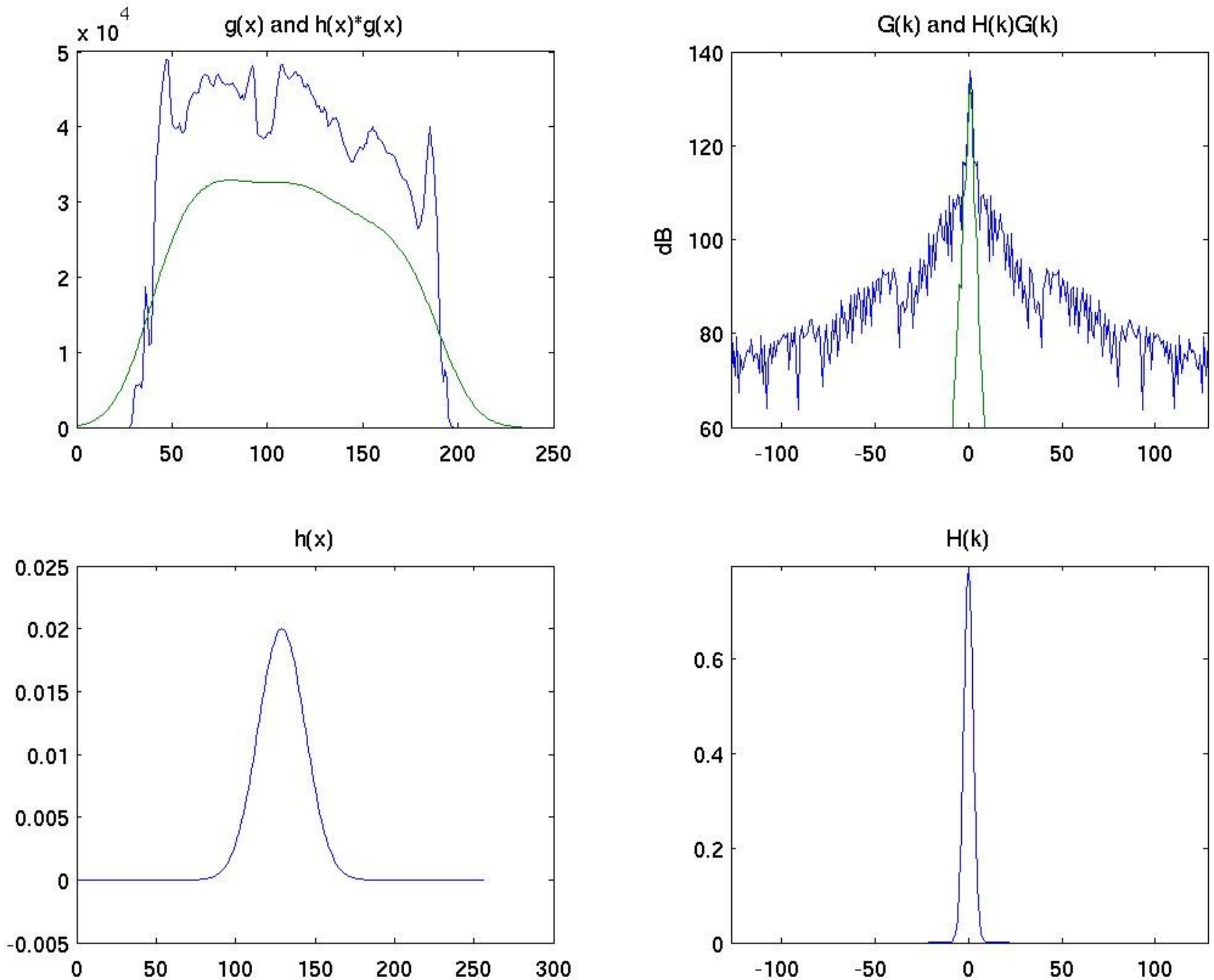
Fourier Domain System Response

Radon inverse filter kernel, $H(k) = |\, k \,|$ (high pass filter)

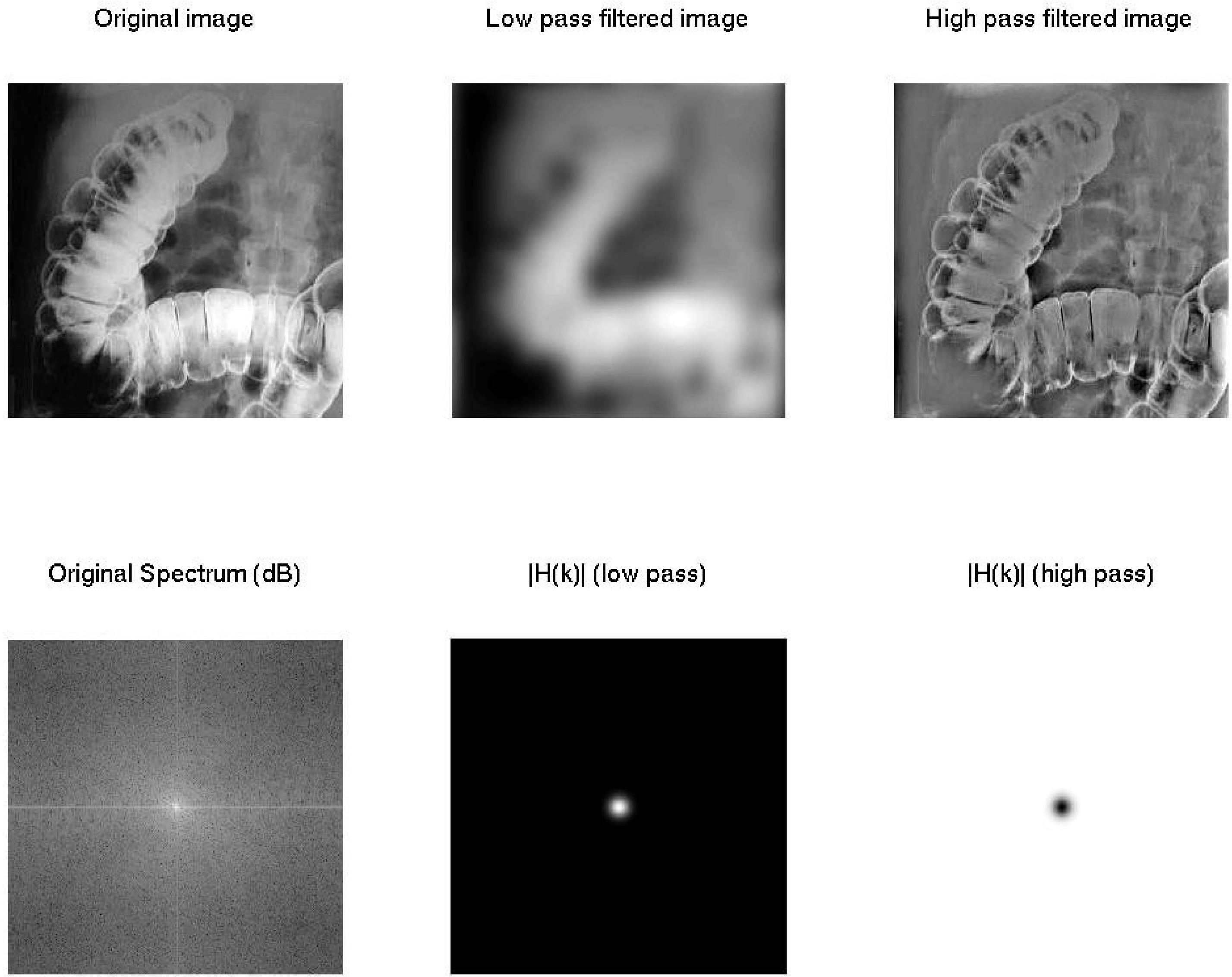


Fourier Domain System Response

Gaussian low-pass filter kernel, $H(k) = \exp\left(\frac{-k^2}{2\sigma^2}\right), \quad \sigma = 1.27$



Fourier Domain System Response 2D



Fourier Transform – FFT

The numerical implementation of the FT is discrete in x and k is referred to as Discrete Fourier Transform (DFT).

$$G[k] = \sum_{x=0}^{N-1} g[x] e^{-j2\pi kx/N}$$

$$g[x] = \frac{1}{N} \sum_{k=0}^{N-1} G[k] e^{j2\pi kx/N}$$

- A fast algorithm is available (FFT) to compute the DFT in only $N \log_2 N$ operations instead of N^2.
- Convolution with long filters is therefore often implemented using the FFT.
- FFT requires N to be a power of 2.

Fourier Transform – Filter with FFT

To filter signals with lengths not a power of 2:

- Pad zeros at the end up to next power of 2
- FFT
- multiply each frequency
- inverse FFT
- keep only fist L values.

Matlab $b(x,y) = h(x,y) * g(x,y)$ using fft (assumes h smaller than g):

```
L = size(g);
N = 2.^ceil(log2(L));
b = ifft2(fft2(g,N(1),N(2)).*fft2(h,N(1),N(2)));
if isreal([g;h]) b = real(b); end
b = b(1:L(1),1:L(2));
```

Demonstrate: `fftshift`, inverse filtering, separability.
Assignment FT: Generate image with randomized phase and original amplitude, and image with random amplitude and original phase.